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The Girth of a Thin Distance-Regular Graph

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Abstract. Let Γ be a distance-regular graph of diameter $d\geq 3$. For each vertex x of Γ, let $T(x)$ denote the Terwilliger algebra for Γ with respect to x. An irreducible $T(x)$-module W is said to be *thin* if $\dim E_i^*(x)W\leq 1$ for $0\leq i\leq d$, where $E_i^*(x)$ is the i^{th} dual idempotent for Γ with respect to x. The graph Γ is *thin* if for each vertex x of Γ, every irreducible $T(x)$-module is thin. A *regular generalized quadrangle* is a bipartite distance-regular graph with girth 8 and diameter 4. Our main results are as follows:

Theorem *Let $\Gamma=(X,R)$ be a distance-regular graph with diameter $d\geq 3$ and valency $k\geq 3$. Then the following are equivalent:*

(i) Γ is a regular generalized quadrangle.

(ii) Γ is thin and $c_3=1$.

Corollary *Let $\Gamma=(X,R)$ be a thin distance-regular graph with diameter $d\geq 3$ and valency $k\geq 3$. Then Γ has girth 3, 4, 6, or 8. The girth of Γ is 8 exactly when Γ is a regular generalized quadrangle.*

Introduction

The purpose of the present paper is to provide an introduction to the results and techniques of [2]. In that paper, we show that if Γ is thin (see definition below), then the girth of Γ is 3, 4, 6, or 8. Moreover, the girth is 8 exactly when Γ is a regular generalized quadrangle.

Let $\Gamma=(X,R)$ be a graph with shortest-path distance function ∂ and diameter d. For any two vertices $x,y\in X$, a *walk of length h from x to y* is a sequence $x_0,x_1,x_2,\ldots,x_h$ ($x_i\in X$, $0\le i\le h$) such that $x_0=x$, $x_h=y$, and x_i is adjacent to x_{i+1} for all i ($0\le i\le h-1$).

A walk in Γ is said to be *closed* if it starts and ends at the same vertex. By a *cycle*, we mean a closed walk $x_0,x_1,x_2,\ldots,x_h=x_0$ of length $h\ge 3$ such that the vertices $x_0,x_1,\ldots,x_{h-1}$ are distinct. The *girth* $g=g(\Gamma)$ is defined to be

$$g = \min\{h \mid \text{there is a cycle of length } h \text{ in } \Gamma\}.$$

Let $\Gamma=(X,R)$ be a graph with diameter d. We say Γ is *regular with valency k* if each vertex in Γ has exactly k neighbors. We say Γ is *distance-regular* whenever for all triples h,i,j ($0\le h,i,j\le d$), and for all $x,y\in X$ with $\partial(x,y) = h$, the number

$$p^h_{ij} = |\{z\in X \mid \partial(x,z)=i,\ \partial(y,z) = j\}|$$

is independent of the choice of x and y. The integers p^h_{ij} are called the *intersection numbers* of Γ.

From now on, assume that Γ is distance-regular. For convenience, set $a_i = p^i_{i1}$ ($0\le i\le d$), $b_i = p^i_{i+1,1}$ ($0\le i\le d-1$), $b_d=0$, $c_i = p^i_{i-1,1}$ ($1\le i\le d$), and $c_0=0$. Note that Γ is regular with valency $k=b_0$. Moreover,

$$k = a_i + b_i + c_i \qquad (0\le i\le d). \tag{1}$$

It can be shown [1, p. 126-127] that $b_0,b_1,\ldots,b_{d-1}$ and $c_1,c_2,\ldots,c_d$ determine $\{p^h_{ij} \mid 0\le h,i,j\le d\}$. The array

$$\left\{\begin{matrix} * & c_1 & c_2 & \cdots & c_d \\ * & a_1 & a_2 & \cdots & a_d \\ b_0 & b_1 & b_2 & \cdots & * \end{matrix}\right\}$$

is known as the *intersection array* of Γ.

Let $\Gamma = (X,R)$ be a distance-regular graph of diameter d. Let $\mathrm{Mat}_X(\mathbb{R})$ denote the $\mathbb{R}$-algebra of matrices with entries in $\mathbb{R}$ and rows and columns indexed by X. For each i $(0 \le i \le d)$, let $A_i = A_i(\Gamma)$ be the matrix in $\mathrm{Mat}_X(\mathbb{R})$ with xy entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y)=i \\ 0 & \text{otherwise} \end{cases} \qquad (x,y \in X).$$

We call A_i the *i^{th} distance matrix of Γ.*

By matrix multiplication, using the definition of the p^h_{ij},

$$A_iA_j = \sum_{h=0}^{d} p^h_{ij}A_h, \qquad (0 \le i,j \le d).$$

Therefore, $\{A_0, A_1, \ldots, A_d\}$ is a basis for a subalgebra $\mathcal{M}$ of $\mathrm{Mat}_X(\mathbb{R})$. We call $\mathcal{M}$ the *Bose-Mesner Algebra of Γ.*

Fix a vertex $x \in X$. For each i $(0 \le i \le d)$, let $E_i^* = E_i^*(x)$ be the diagonal matrix in $\mathrm{Mat}_X(\mathbb{R})$ with yy entry

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x,y)=i \\ 0 & \text{otherwise} \end{cases} \qquad (y \in X).$$

We call E_i^* the *i^{th} dual idempotent of Γ with respect to x.*

Observe that

$$E_i^*E_j^* = \delta_{ij}E_i^*, \qquad (0 \le i,j \le d).$$

Therefore, $\{E_0^*, E_1^*, \ldots, E_d^*\}$ is a basis for a subalgebra $\mathcal{M}^* = \mathcal{M}^*(x)$ of $\mathrm{Mat}_X(\mathbb{R})$. We call $\mathcal{M}^*$ the *dual Bose-Mesner Algebra of Γ with respect to x.*

Let $\Gamma = (X,R)$ be a distance-regular graph. Pick $x \in X$, and write $\mathcal{M}^* = \mathcal{M}^*(x)$. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{R})$ generated by $\mathcal{M}$ and $\mathcal{M}^*$. We refer to T as the *Terwilliger Algebra of Γ with respect to x.*

Let $\Gamma=(X,R)$ be a distance-regular graph of diameter d. Let $V = \mathbb{R}^X$ denote the column space of $Mat_X(\mathbb{R})$. Then $Mat_X(\mathbb{R})$ acts on V by left multiplication. We refer to V as the *standard module* for Γ. Fix a vertex $x\in X$, and write $T=T(x)$, and $E_i^*=E_i^*(x)$ $(0\le i\le d)$. By a *T-module*, we mean a subspace of the standard module V which is invariant under multiplication by elements of T. A nonzero T-module W is said to be *irreducible* if W properly contains no T-modules other than 0. An irreducible T-module W is said to be *thin* if

$$\dim E_i^*W \le 1 \qquad (0\le i\le d).$$

For all $x\in X$, we say Γ is *thin with respect to x* whenever every irreducible T(x)-module is thin. We say Γ is *thin* if Γ is thin with respect to every vertex $x\in X$.

The Terwilliger Algebra in general, and thin graphs in particular, have been studied extensively in recent years. See, for example, [3], [4], [5] .

A Combinatorial Interpretation of the Thin Condition

We say that a walk $y_0,y_1,\ldots,y_h$ in Γ has *shape* $i_0,i_1,\ldots,i_h$ *with respect to x* if $\partial(x,y_j)=i_j$ for all j $(0\le j\le h)$.

In Figure 1, the bubble labeled $\Gamma_1(x)$ represents the set of vertices adjacent to x, the bubble labeled $\Gamma_2(x)$ represents the set of vertices distance 2 from x, etc. Thus, the pictured path from y to z has shape 4,4,3,2,3,3,4 with respect to x.

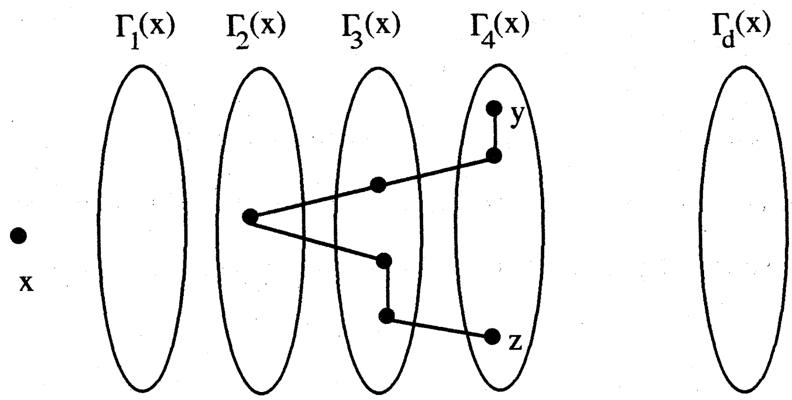

Figure 1

The following Lemma gives a combinatorial interpretation of the thin condition in terms of the shape of paths.

Lemma 1 *Suppose $\Gamma=(X,R)$ is a distance-regular graph with diameter $d\geq 3$. Pick $x\in X$, and write $E_i^*=E_i^*(x)$ $(0\leq i\leq d)$. The following are equivalent:*

(i) Γ is thin with respect to x.

(ii) For any integer i $(0\leq i\leq d)$, for any sequence of integers $i=i_0,i_1,\ldots,i_h=i$ $(0\leq i_j\leq d,\ 0\leq j\leq h)$, and for any vertices y,z at distance i from x, the number of walks from y to z of shape $i_0,i_1,\ldots,i_h$ with respect to x is equal to the number of walks from z to y of shape $i_0,\ i_1,\ldots,i_h$ with respect to x.

For our purposes, the important implication is (i)⇒(ii). If a distance-regular graph is thin, then the existence of a path from y to z of a certain shape implies the existence of a path from z to y of the same shape. For example, assuming the graph in Figure 1 is thin, the existence of the pictured path implies the existence of the path in Figure 2.

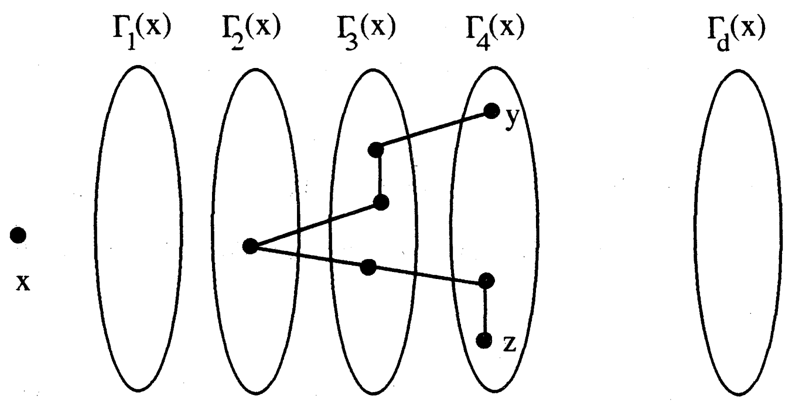

Figure 2

We can already see how this Lemma will be used to produce a girth bound. The two paths together imply the existence of a cycle of length at most 12. Our strategy will be to use various techniques to produce short, asymmetric paths, which will imply the existence of short cycles.

A Characterization of the Regular Generalize Quadrangles

By a *regular generalized quadrangle*, we mean a distance-regular graph with intersection array:

$$\begin{Bmatrix} * & 1 & 1 & 1 & k \\ * & 0 & 0 & 0 & 0 \\ k & k-1 & k-1 & k-1 & * \end{Bmatrix}.$$

The main result of our paper is the following.

Theorem 2 *Let $\Gamma=(X,R)$ be a distance-regular graph with diameter $d\geq 3$ and valency $k\geq 3$. Then the following are equivalent:*

(i) Γ is a regular generalized quadrangle.

(ii) Γ is thin and $c_3=1$.

The outline of the proof is as follows:

Step 1: Show the implication (i)⇒(ii).

Step 2: Assume (ii), and show that $g>6$.

Step 3: Show that g is even.

Step 4: Show that $g=8$.

Step 5: Show that $c_4=k$, which implies that Γ is a generalized quadrangle.

Space limitations do not permit us to show the entire proof. However, we will illustrate some of the important ideas involved by presenting some details of steps 3, 4, and 5.

Step 3: Show that g is even.

Suppose that g=2i+1 for some integer i (i≥3). Pick a cycle $x_0,x_1,\ldots,x_{2i+1}=x_0$ of minimal length. Note that since this cycle has minimal length, $\partial(x_0,x_{i-1})=i-1$ and $\partial(x_0,x_i)=\partial(x_0,x_{i+1})=i$. Now, since k≥3, x_{i-1} must have a neighbor y that is not part of the cycle. Since the girth is 2i+1, $\partial(x,y)=i$. See Figure 3.

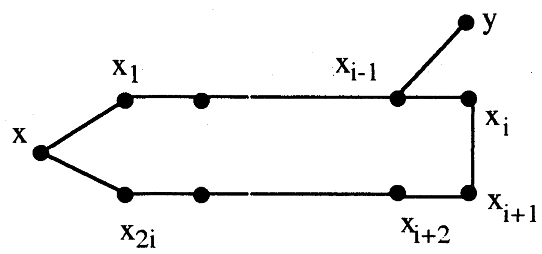

Figure 3

Now, the existence of the path y,x_{i-1},x_i,x_{i+1} of shape i,i-1,i,i with respect to x implies the existence of a path from x_{i+1} to y with the same shape with respect to x. But the two paths together form a cycle of length at most 6, a contradiction.

Step 4: Show that g=8.

Suppose that g=2i for some integer i (i≥4). Pick a cycle $x_0,x_1,\ldots,x_{2i}=x_0$ of minimal length. Note that since this cycle has minimal length, $\partial(x_0,x_{i-2})=i-2$, $\partial(x_0,x_{i-1})=\partial(x_0,x_{i+1})=i-1$, and $\partial(x_0,x_i)=i$. Now, since k≥3, x_{i-2} must have a neighbor y that is not part of the cycle. Since the girth is 2i, $\partial(x,y)=i-1$. See Figure 4.

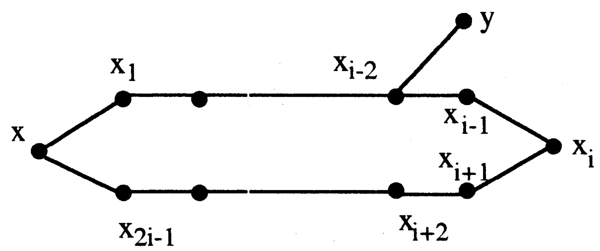

Figure 4

Now, the existence of the path $y,x_{i-2},x_{i-1},x_i,x_{i+1}$ of shape i-1,i-2,i-1,i,i-1 with respect to x implies the existence of a path from x_{i+1} to y with the same shape with respect to x. The two paths together form a cycle of length at most 8. Therefore, the girth of Γ is 8.

We now know that the intersection array of Γ is:

$$\begin{Bmatrix} * & 1 & 1 & 1 & ? & ? & \dots \\ 0 & 0 & 0 & 0 & ? & ? & \dots \\ k & k-1 & k-1 & k-1 & ? & ? & \dots \end{Bmatrix}.$$

It will therefore be sufficient to establish that c_4=k. By (1), this will imply that a_4=b_4=0.

Step 5: Show c_4 = k.

Pick a cycle $x_0,x_1,\dots,x_8=x_0$. Let y be any neighbor of x_4. We wish to show that $\partial(x_0,y)=3$. Of course, we may assume that $y \neq x_3,x_5$. Thus, we have the following picture:

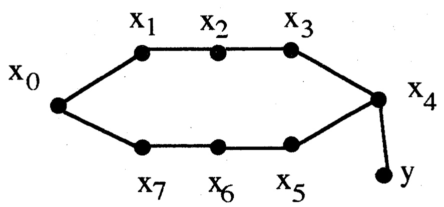

Figure 5

Now, look at the shape of the path y,x_4,x_5,x_6,x_7 **with respect to x_2**. With respect to x_2, this path has shape 3,2,3,4,3. Therefore, there must be a path from x_7 to y with shape 3,2,3,4,3. The second vertex in this path is adjacent to x_7 and distance 2 from x_2. Since c_3 = 1, the unique vertex of this description in x_0. Therefore, there is a path from x_0 to y of length 3, and $\partial(x_0,y)=3$, as desired.

This completes the proof of Theorem 2. We have the following immediate Corollary.

Corollary 3 *Let $\Gamma=(X,R)$ be a thin distance-regular graph with diameter $d\geq 3$ and valency $k\geq 3$. Then Γ has girth 3, 4, 6, or 8. The girth of Γ is 8 exactly when Γ is a regular generalized quadrangle.*

Proof If $c_3>1$, there is a cycle of length 6, so $g\leq 6$. (The fact that $g\neq 5$ follows from some of the details we omitted in Step 1.) If $c_3=1$, then $g=8$ by Theorem 2.

References

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